Jointly Distributed Random Variables

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Joint Distribution Functions

Thus far, we have concerned ourselves only with probability distributions for single random variables. However, we are often interested in probability statements concerning two or more random variables. In order to deal with such probabilities, we define, for any two random variables X and Y, the *joint cumulative probability distribution function* of X and Y by

$$F(a, b) = P\{X \leq a, Y \leq b\} = P(\{X \leq a\} \cap \{Y \leq b\}) \quad -\infty < a, b < \infty.$$

The distribution of X can be obtained from the joint distribution of X and Y as follows:

$$\begin{aligned} F_X(a) &= P\{X \leq a\} \\ &= P\{X \leq a, Y < \infty\} \\ &= P\left(\lim_{b \to \infty} \{X \leq a, Y \leq b\}\right) \\ &= \lim_{b \to \infty} P\{X \leq a, Y \leq b\} \\ &= \lim_{b \to \infty} F(a, b) \\ &\equiv F(a, \infty). \end{aligned}$$

Joint Distribution Functions

Note that, in the preceding set of equalities, we have once again made use of the fact that probability is a continuous set (that is, event) function. Similarly, the cumulative distribution function of Y is given by

$$\begin{aligned} F_Y(b) &= P\{Y \leq b\} \\ &= \lim_{a \to \infty} F(a, b) \\ &\equiv F(\infty, b). \end{aligned}$$

The distribution functions F_X and F_Y are sometimes referred to as the *marginal* distributions of X and Y.

All joint probability statements about X and Y can, in theory, be answered in terms of their joint distribution function. For instance, suppose we wanted to compute the joint probability that X is greater than a and Y is greater than b. This could be done as follows:

$$\begin{aligned} P\{X > a, Y > b\} &= 1 - P(\{X > a, Y > b\}^c) \\ &= 1 - P(\{X > a\}^c \cup \{Y > b\}^c) \\ &= 1 - P(\{X \leq a\} \cup \{Y \leq b\}) \\ &= 1 - [P\{X \leq a\} + P\{Y \leq b\} - P\{X \leq a, Y \leq b\}] \\ &= 1 - F_X(a) - F_Y(b) + F(a, b). \end{aligned} \tag{1}$$

Joint Distribution Functions

Equation (1) is a special case of the following equation, whose verification is left as an exercise:

$$\begin{aligned} &P\{a_1 < X \leq a_2, b_1 < Y \leq b_2\} \\ &= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1) \end{aligned} \tag{2}$$

whenever $a_1 < a_2$, $b_1 < b_2$.
In the case when X and Y are both discrete random variables, it is convenient to define the *joint probability mass function* of X and Y by

$$p(x, y) = P\{X = x, Y = y\}.$$

The probability mass function of X can be obtained from $p(x, y)$ by

$$\begin{aligned} p_X(x) &= P\{X = x\} \\ &= \sum_{y:p(x,y)>0} p(x, y). \end{aligned}$$

Similarly,

$$p_Y(y) = \sum_{x:p(x,y)>0} p(x, y).$$

Example 1.

Suppose that 3 *balls are randomly selected from an urn containing* 3 *red,* 4 *white, and* 5 *blue balls. If we let* X *and* Y *denote, respectively, the number of red and white balls chosen, then the joint probability mass function of* X *and* Y, $p(i,j) = P\{X = i, Y = j\}$, *is given by*

$$p(0,0) = \binom{5}{3} \Big/ \binom{12}{3} = \frac{10}{220}$$

$$p(0,1) = \binom{4}{1}\binom{5}{2} \Big/ \binom{12}{3} = \frac{40}{220}$$

$$p(0,2) = \binom{4}{2}\binom{5}{1} \Big/ \binom{12}{3} = \frac{30}{220}$$

$$p(0,3) = \binom{4}{3} \Big/ \binom{12}{3} = \frac{4}{220}$$

$$p(1,0) = \binom{3}{1}\binom{5}{2} \Big/ \binom{12}{3} = \frac{30}{220}$$

$$p(1,1) = \binom{3}{1}\binom{4}{1}\binom{5}{1} \Big/ \binom{12}{3} = \frac{60}{220}$$

$$p(1,2) = \binom{3}{1}\binom{4}{2} \Big/ \binom{12}{3} = \frac{18}{220}$$

Example (Contd...)

$$p(2,0) = \binom{3}{2}\binom{5}{1} \Big/ \binom{12}{3} = \frac{15}{220}$$
$$p(2,1) = \binom{3}{2}\binom{4}{1} \Big/ \binom{12}{3} = \frac{12}{220}$$
$$p(3,0) = \binom{3}{3} \Big/ \binom{12}{3} = \frac{1}{220}$$

These probabilities can most easily be expressed in tabular form, as in Table 6.1. The reader should note that the probability mass function of X is obtained by computing the row sums, whereas the probability mass function of Y is obtained by computing the column sums. Because the individual probability mass functions of X and Y thus appear in the margin of such a table, they are often referred to as the *marginal probability mass functions* of X and Y, respectively.

i \ j	0	1	2	3	Row sum $= P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column sum $= P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Example

Example 2.

Suppose that 15 *percent of the families in a certain community have no children,* 20 *percent have* 1 *child,* 35 *percent have* 2 *children, and* 30 *percent have* 3*. Suppose further that in each family each child is equally likely (independently) to be a boy or a girl. If a family is chosen at random from this community, then B, the number of boys, and G, the number of girls, in this family will have the joint probability mass function shown in Table 6.2.*

i \ j	0	1	2	3	Row sum $= P\{B = i\}$
0	.15	.10	.0875	.0375	.3750
1	.10	.175	.1125	0	.3875
2	.0875	.1125	0	0	.2000
3	.0375	0	0	0	.0375
Column sum $= P\{G = j\}$	.3750	.3875	.2000	.0375	

Example (Contd...)

The probabilities shown in the Table above are obtained as follows:

$$
\begin{aligned}
P\{B=0, G=0\} &= P\{\text{no children}\} = .15 \\
P\{B=0, G=1\} &= P\{\text{1 girl and total of 1 child}\} \\
&= P\{\text{1 child}\}P\{\text{1 girl}|\text{1 child}\} = (.20)\left(\frac{1}{2}\right) \\
P\{B=0, G=2\} &= P\{\text{2 girls and total of 2 children}\} \\
&= P\{\text{2 children}\}P\{\text{2 girls}|\text{2 children}\} = (.35)\left(\frac{1}{2}\right)^2.
\end{aligned}
$$

Example (Contd...)

We say that X and Y are *jointly continuous* if there exists a function $f(x, y)$, defined for all real x and y, having the property that, for every set C of pairs of real numbers (that is, C is a set in the two-dimensional plane),

$$P\{(X, Y) \in C\} = \iint_{(x,y)\in C} f(x, y)dx\,dy. \tag{3}$$

The function $f(x, y)$ is called the *joint probability density* function of X and Y. If A and B are any sets of real numbers, then, by defining $C = \{(x, y) : x \in A, y \in B\}$, we see from Equation (3) that

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x, y)dx\,dy. \tag{4}$$

Because

$$\begin{aligned} F(a, b) &= P\{X \in (-\infty, a], Y \in (-\infty, b]\} \\ &= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)dx\,dy \end{aligned}$$

it follows, upon differentiation, that

$$f(a, b) = \frac{\partial^2}{\partial a \partial b} F(a, b)$$

wherever the partial derivatives are defined. Another interpretation of the joint density function, obtained from Equation (4), is

Example (Contd...)

$$P\{a < X < a + da, b < Y < b + db\} = \int_b^{d+db} \int_a^{a+da} f(x, y) dxdy$$
$$\approx f(a, b) dadb$$

when da and db are small and $f(x, y)$ is continuous at a, b. Hence, $f(a, b)$ is a measure of how likely it is that the random vector (X, Y) will be near (a, b).
If X and Y are jointly continuous, they are individually continuous, and their probability density functions can be obtained as follows:

$$\begin{aligned} P\{X \in A\} &= P\{X \in A, Y \in (-\infty, \infty)\} \\ &= \int_A \int_{-\infty}^{\infty} f(x, y) dydx \\ &= \int_A f_X(x) dx \end{aligned}$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

is thus the probability density function of X. Similarly, the probability density function of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

Example 3.

The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \textit{otherwise} \end{cases}$$

Compute (a) $P\{X > 1, Y < 1\}$, (b) $P\{X < Y\}$, and (c) $P\{X < a\}$.

Solution: (a)

$$\begin{aligned} P\{X > 1, Y < 1\} &= \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dxdy \\ &= \int_0^1 2e^{-2y}(-e^{-x}|_1^\infty)dy \\ &= e^{-1}\int_0^1 2e^{-2y}\,dy \\ &= e^{-1}(1 - e^{-2}). \end{aligned}$$

Solution (Contd...)

(b)

$$
\begin{aligned}
P\{X < Y\} &= \iint_{(x,y):x<y} 2e^{-x}e^{-2y}\,dxdy \\
&= \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dxdy \\
&= \int_0^\infty 2e^{-2y}(1 - e^{-y})dy \\
&= \int_0^\infty 2e^{-2y}dy - \int_0^\infty 2e^{-3y}dy \\
&= 1 - \frac{2}{3} \\
&= \frac{1}{3}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
P\{X < a\} &= \int_0^a \int_0^\infty 2e^{-2y}e^{-x}\,dydx \\
&= \int_0^a e^{-x}dx \\
&= 1 - e^{-a}.
\end{aligned}
$$

Example 4.

Consider a circle of radius R, and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen (Figure 6.1), then, since (X, Y) is equally likely to be near each point in the circle, it follows that the joint density function of X and Y is given by

$$f(x, y) = \begin{cases} c & \text{if } x^2 + y^2 \leq R^2 \\ 0 & \text{if } x^2 + y^2 > R^2 \end{cases}$$

for some value of c.

(a) *Determine c.*

(b) *Find the marginal density functions of X and Y.*

(c) *Compute the probability that D, the distance from the origin of the point selected, is less than or equal to a.*

(d) *Find $E[D]$.*

Figure 6.1: Joint probability distribution.

Solution

(a) Because

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$$

it follows that

$$c \iint\limits_{x^2+y^2\leq R^2} dydx = 1.$$

We can evaluate $\iint_{x^2+y^2\leq R^2} dydx$ either by using polar coordinates or, more simply, by noting that it represents the area of the circle and is thus equal to πR^2. Hence,

$$c = \frac{1}{\pi R^2}.$$

Solution (Contd...)

(b)

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \frac{1}{\pi R^2} \int_{x^2+y^2 \leq R^2} dy \\
&= \frac{1}{\pi R^2} \int_{-c}^{c} dy, \quad \text{where } c = \sqrt{R^2 - x^2} \\
&= \frac{2}{\pi R^2} \sqrt{R^2 - x^2} \quad x^2 \leq R^2
\end{aligned}
$$

and it equals 0 when $x^2 > R^2$. By symmetry, the marginal density of Y is given by

$$
\begin{aligned}
f_Y(y) &= \frac{2}{\pi R^2} \sqrt{R^2 - y^2} & y^2 \leq R^2 \\
&= 0 & y^2 > R^2.
\end{aligned}
$$

Solution (Contd...)

(c) The distribution function of $D = \sqrt{X^2 + Y^2}$, the distance from the origin, is obtained as follows: For $0 \leq a \leq R$,

$$\begin{aligned} F_D(a) &= P\{\sqrt{X^2 + Y^2} \leq a\} \\ &= P\{X^2 + Y^2 \leq a^2\} \\ &= \iint\limits_{x^2+y^2 \leq a^2} f(x, y)dydx \\ &= \frac{1}{\pi R^2} \iint\limits_{x^2+y^2 \leq a^2} dydx \\ &= \frac{\pi a^2}{\pi R^2} \\ &= \frac{a^2}{R^2} \end{aligned}$$

where we have used the fact that $\iint_{x^2+y^2 \leq a^2} dydx$ is the area of a circle of radius a and thus is equal to πa^2.

Solution (Contd...)

(d) From part (c), the density function of D is

$$f_D(a) = \frac{2a}{R^2} \quad 0 \leq a \leq R.$$

Hence,

$$E[D] = \frac{2}{R^2} \int_0^R a^2 da = \frac{2R}{3}.$$

Example 5.

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \textit{otherwise} \end{cases}$$

Find the density function of the random variable X/Y.

Solution. *We start by computing the distribution function of X/Y. For $a > 0$,*

$$F_{X/Y}(a) = \left\{ \frac{X}{Y} \le a \right\}$$
$$= \iint\limits_{x/y \le a} e^{-(+y)} dx\ dy$$

Example 1 Continued

$$
\begin{aligned}
&= \int_0^\infty \int_0^{ay} e^{-(x+y)} dx \ dy \\
&= \int_0^\infty (1 - e^{-ay}) e^{-y} dy \\
&= \left\{ -e^{-y} + \frac{e^{-(a+1)y}}{a+1} \right\} \Bigg|_0^\infty \\
&= 1 - \frac{1}{a+1}
\end{aligned}
$$

Differentiation shows that the density function of X/Y is given by $f_{X/Y}(a) = 1/(a+1)^2, 0 < a < \infty$.

We can also define joint probability distributions for n random variables in exactly the same manner as we did for $n = 2$. For instance, the joint cumulative probability distribution function $F(a_1, a_2, \cdots, a_n)$ of the n

Example 1 Continued

random variables $X_1, X_2, \cdots, X_n$ is defined by

$$F(a_1, a_2, \cdots, a_n) = P\{X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n\}$$

Further, the n random variables are said to be jointly continuous if there exists a function $f(x_1, x_2, ..., x_n)$, called the joint probability density function, such that, for any set C in n-space,

$$P\{(X_1, X_2, \ldots, X_n) \in C\} = \iint\limits_{(x_1,\ldots,x_n)\in C} \cdots \int f(x_1, \ldots, x_n) dx_1 dx_2 \cdots dx_n$$

In particular, for any n sets of real numbers $A_1, A_2, \ldots, A_n$,

$$P\{X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\}$$

$$= \int_{A_n} \int_{A_{n-1}} \cdots \int_{A_1} f(x_1, \ldots, x_n) dx_1 dx_2 \cdots dx_n$$

Example 6.

The Multinomial distribution

One of the most important joint distributions is the multinomial distribution, which arises when a sequence of n independent and identical experiments is performed. Suppose that each experiment can result in any one of r possible outcomes, with respective probabilities $p_1, p_2, \ldots, p_r, \sum_{i=1}^{r} p_i = 1$. If we let X_i denote the number of the n experiments that result in outcome number i, then

$$P\{X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r\} = \frac{n!}{n_1!n_2!\cdots n_r!} p_1^{n_1} p_2^{n_1} \cdots p_r^{n_r} \qquad (5)$$

whenever $\sum_{i=1}^{r} n_i = n$.
Equation 5 is verified by noting that any sequence of outcomes for the n experiments that leads to outcome i occurring n_i times for $i = 1, 2, \ldots, r$ will, by the assumed independence of experiments, have probability $p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ of occurring. Because there are $n!/(n_1!n_2!...n_r!)$ such

Example 2 Continued

sequences of outcomes (there are $n!/n_1! \dots n_r!$ different permutations of n things of which n_1 are alike, n_2 are alike,$\dots$, n_r are alike), Equation 5 is established. The joint distribution whose joint probability mass function is specified by Equation 5 is called the multinomial distribution. Note that when $r = 2$, the multinomial reduces to the binomial distribution.

Note also that any sum of a fixed set of theX_s^i will have a binomial distribution. That is, if $N(\{1, 2, \dots, r\}$, then $\sum_{i \in N} X_i$ will be a binomial random variable with parameters n and $p = \sum_{i \in N} pi$. This follows because $\sum_{i \in N} X_i$ represents the number of the n experiments whose outcome is in N, and each experiment will independently have such an outcome with probability $\sum_{i \in N} p_i$.

As an application of the multinomial distribution, suppose that a fair die is rolled 9 times. The probability that 1 appears three times, 2 and 3 twice each, 4 and 5 once each, and 6 not at all is

The multinomial distribution

$$\frac{9!}{3!2!2!1!1!0!}\left(\frac{1}{6}\right)^3\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{2}\right)^0=\frac{9!}{3!2!2!}\left(\frac{1}{6}\right)^9$$

Independent Random Variables

The random variables X and Y are said to be independent if, for any two sets of real numbers A and B,

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\} \tag{6}$$

In other words, X and Y are independent if, for all A and B, the events $E_A = \{X \in A\}$ and $F_B = \{Y \in B\}$ are independent.

It can be shown by using the three axioms of probability that Equation 6 will follow if and only if, for all a, b,

$$P\{X \leq a, Y \leq b\} = P\{X \leq a\}P\{Y \leq b\}$$

Hence, in terms of the joint distribution function F of X and Y, X and Y are independent if

$$F(a, b) = F_x(a)F_Y(b) \quad \text{for all } a, b.$$

Independent Random Variables

When X and Y are discrete random variables, the condition of independence 6 is equivalent to

$$p(x,y) = p_X(x)py(y) \quad \textit{forall } x,y \tag{7}$$

The equivalence follows because, if Equation 6 is satisfied, then we obtain Equation 7 by letting A and B be, respectively, the one-point sets $A = \{x\}$ and $B = \{y\}$. Furthermore, if Equation 7 is valid, then, for any sets A, B,

$$\begin{aligned} P\{X \in A, Y \in B\} &= \sum_{y \in B} \sum_{x \in A} p(x,y) \\ &= \sum_{y \in B} \sum_{x \in A} px(x)py(y) \\ &= \sum_{y \in B} p_Y(y) \sum_{x \in A} px(x) \\ &= P\{Y \in B\} P\{x \in A\} \end{aligned}$$

and Equation 6 is established.

Independent Random Variables

In the jointly continuous case, the condition of independence is equivalent to

$$f(x, y) = f_X(x) f_Y(y) \quad \text{for all } x, y$$

Thus, loosely speaking, X and Y are independent if knowing the value of one does not change the distribution of the other. Random variables that are not independent are said to be dependent.

Example

Example 7.

Suppose that $n + m$ independent trials having a common probability of success p are performed. If X is the number of successes in the first n trials, and Y is the number of successes in the final m trials, then X and Y are independent, since knowing the number of successes in the first n trials does not affect the distribution of the number of successes in the final m trials (by the assumption of independent trials). In fact, for integral x and y,

$$\begin{aligned} P\{X = x, Y = y\} &= \binom{n}{x} p^x (1-p)^{n-x} \binom{m}{y} p^y (1-p)^{m-y} \quad \begin{matrix} 0 \le x \le n, \\ 0 \le y \le m \end{matrix} \\ &= P\{X = x\} P\{Y = y\} \end{aligned}$$

In contrast, X and Z will be dependent, where Z is the total number of successes in the $n + m$ trials. (Why?) .

Example 8.

Suppose that the number of people who enter a post office on a given day is a Poisson random variable with parameter λ. Show that if each person who enters the post office is a male with probability p and a female with probability $1 - p$, then the number of males and females entering the post office are independent Poisson random variables with respective parameters λp and $\lambda(1 - p)$.

Example 9.

A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 P.M., find the probability that the first to arrive has to wait longer than 10 minutes.

Solution. *If we let X and Y denote, respectively, the time past 12 that the man and the woman arrive, then X and Y are independent random variables, each of which is uniformly distributed over $(0, 60)$. The desired probability, $P\{X + 10 < Y\} + P\{Y + 10 < X\}$, which, by symmetry, equals $2P\{X + 10 < Y\}$, is obtained as follows:*

$$2P\{X + 10 < Y\} = 2 \iint_{x+10<y} f(x, y)dx\, dy$$

$$= 2 \iint_{x+10<y} f_X(x) f_Y \, dx \; dy$$

$$= 2 \int_{10}^{60} \int_{0}^{y-10} \left(\frac{1}{60}\right)^2 dx \; dy$$

$$= \frac{2}{(60)^2} \int_{10}^{60} (y - 10) dy$$

$$= \frac{25}{36}$$

Our next example presents the oldest problem dealing with geometrical probabilities. It was first considered and solved by Buffon, a French naturalist of the 18th century, and is usually referred to as Buffon's needle problem.

Example

Example 10.

Buffon's needle problem

A table is ruled with equidistant parallel lines a distance D apart.A needle of length L, where $L \leq D$, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)?

Solution. *Let us determine the position of the needle by specifying* (1) *the distance X from the middle point of the needle to the nearest parallel line and* (2) *the angle* 0 *between the needle and the projected line of length X. (See Figure* 6.2.*) The needle will intersect a line if the hypotenuse of the right triangle in Figure* 6.2 *is less than* $L/2$ *- that is, if*

$$\frac{X}{\cos\theta} < \frac{L}{2} \text{ or } X < \frac{L}{2}\cos\theta$$

image

Conditional Distributions: Continuous Case

If X and Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X given that $Y = y$ is defined, for all values of y such that $f_Y(y) > 0$, by

$$f_{X|Y}(x|y)dx = \frac{f(x, y)}{f_Y(y)}$$

To motivate this definition, multiply the left-hand side by dx and the right-hand side by $(dx\ dy)/dy$ to obtain

$$\begin{aligned} f_{x|y}(x|y)dx &= \frac{f(x, y)dx\ dy}{f_Y(y)dy} \\ &\approx \frac{P\{x \leq X \leq x + dx, y \leq Y \leq y + dy\}}{P\{y \leq Y \leq y + dy\}} \\ &= P\{x \leq X \leq x + dx | y \leq Y \leq y + dy\} \end{aligned}$$

Conditional Distributions: Continuous Case

In other words, for small values of dx and dy, $f_{X|Y}(x|y)dx$ represents the conditional probability that X is between x and $x + dx$ given that Y is between y and $y + dy$.

The use of conditional densities allows us to define conditional probabilities of events associated with one random variable when we are given the value of a second random variable. That is, if X and Y are jointly continuous, then, for any set A,

$$P\{X \in A | Y = y\} = \int_A f_{X|Y}(x|y)dx$$

In particular, by letting $A = (-\infty, a]$, we can define the conditional cumulative distribution function of X given that $Y = y$ by

Conditional Distributions: Continuous Case

$$F_{X|Y}(a|y) \equiv P\{X \leq a|Y = y\} = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

The reader should note that, by using the ideas presented in the preceding discussion, we have been able to give workable expressions for conditional probabilities, even though the event on which we are conditioning (namely, the event $\{Y = y\}$) has probability 0.

Example 11.

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise \end{cases}$$

Compute the conditional density of X given that $Y = y$, where $0 < y < 1$.
Solution. *For $0 < x < 1, 0 < y < 1$, we have*

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dx} \\ &= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx} \end{aligned}$$

Example 1 Continued

$$= \frac{x(2-x-y)}{\frac{2}{3} - y/2}$$

$$= \frac{6x(2-x-y)}{4-3y}$$

Example 12.

Suppose that the joint density of X *and* Y *is given by*

$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \textit{otherwise} \end{cases}$$

Find $P\{X > 1|Y = y\}$.

Solution. *We first obtain the conditional density of* X *given that* $Y = y$.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{e^{-x/y}e^{-y}/y}{e^{-y}\int_0^\infty (1/y)e^{-x/y}\,dx} \\ &= \frac{1}{y}e^{-x/y}. \end{aligned}$$

Example 2 Continued

Hence,

$$P\{X > 1|Y = y\} = \int_1^\infty \frac{1}{y} e^{-x/y} dx$$
$$= -e^{-x/y}\Big|_1^\infty$$
$$= e^{-1/y}.$$

If X and Y are independent continuous random variables, the conditional density of X given that $Y = y$ is just the unconditional density of X. This is so because, in the independent case,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

We can also talk about conditional distributions when the random variables are neither jointly continuous nor jointly discrete. For example, suppose that X is a continuous random variable having probability density function f and N is a discrete random variable, and consider the conditional distribution of X given that $N = n$. Then

$$\frac{P\{x < X < x + dx | N = n\}}{dx} = \frac{P\{N = n | x < X < x + dx\}}{P\{N = n\}} \frac{P\{x < X < x + dx\}}{dx}$$

and letting dx approach 0 gives

$$\lim_{dx \to 0} \frac{P\{x < X, x + dx | N = n\}}{dx} = \frac{P\{N = n | X = x\}}{P\{N = n\}} f(x)$$

thus showing that the conditional density of X given that $N = n$ is given by

$$f_{X|N}(x|n) = \frac{P\{N = n | X = x\}}{P\{N = n\}} f(x)$$

Joint Probability Distribution of Functions of Random Variables

Let X_1 and X_2 be jointly continuous random variables with joint probability density function f_{X_1,X_2} . It is sometimes necessary to obtain the joint distribution of the random variables Y_1 and Y_2, which arise as functions of X_1 and X_2. Specifically, suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ for some functions g_1 and g_2.

Assume that the functions g_1 and g_2 satisfy the following conditions:

1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2, with solutions given by, say, $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$.
2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the 2×2 determinant

Joint Probability Distribution of Functions of Random Variables

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

at all points (x_1, x_2).

Under these two conditions, it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2)|J(x_1, x_2)|^{-1} \tag{8}$$

where $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$.

A proof of Equation 8 would proceed along the following lines:

Joint Probability Distribution of Functions of Random Variables

$$P\{Y_1 \leq y_1, Y_2 \leq y_2\} = \iint\limits_{\substack{(x_1,x_2): \\ g_1(x_1,x_2)\leq y_1 \\ g_2(x_1,x_2)\leq y_2}} f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \tag{9}$$

The joint density function can now be obtained by differentiating Equation 9 with respect to y_1 and y_2. That the result of this differentiation will be equal to the righthand side of Equation 8 is an exercise in advanced calculus whose proof will not be presented in this book.

Example 13.

Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2}.

Solution. Let $g_1(x_1, x_2) = x_1 + x_2$ and $g_2(x_1, x_2) = x_1 - x_2$. Then

$$J(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Also, since the equations $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ have $x_1 = (y_1 + y_2)/2, x_2 = (y_1 - y_2)/2$ as their solution, it follows from Equation 8 that the desired density is

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1,X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right).$$

Example 3 Continued

For instance, if X_1 and X_2 are independent uniform $(0, 1)$ random variables, then

$$f_{Y_1,Y_2} = \begin{cases} \frac{1}{2} & 0 \leq y_1 + y_2 \leq 2, 0 \leq y_1 - y_2 \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

or if X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, then

$$f_{Y_1,Y_2}(y_1, y_2)$$

$$= \begin{cases} \frac{\lambda_1 \lambda_2}{2} \exp\left\{ -\lambda_1 \left(\frac{y_1+y_2}{2} \right) - \lambda_2 \left(\frac{y_1-y_2}{2} \right) \right\} & y_1 + y_2 \geq 0, y_1 - y_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

image

Figure 6.4 : $\cdot$ = Random point.$(x, y) = (R, \Theta)$.

Example 3 Continued

Finally, if X_1 and X_2 are independent standard normal random variables, then

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \frac{1}{4\pi} e^{-[(y_1+y_2)^2/8+(y_1-y_2)^2/8]} \\ &= \frac{1}{4\pi} e^{-(y_1^2+y_2^2)/4} \\ &= \frac{1}{\sqrt{4\pi}} e^{-y_1^2/4} \frac{1}{\sqrt{4\pi}} e^{-y_2^2/4}. \end{aligned}$$

Thus, not only do we obtain (in agreement with Proposition 3.2) that both $X_1 + X_2$ and $X_1 - X_2$ are normal with mean 0 and variance 2, but we also conclude that these two random variables are independent. (In fact, it can be shown that if X_1 and X_2 are independent random variables having a common distribution function F, then $X_1 + X_2$ will be independent of $X_1 - X_2$ if and only if F is a normal distribution function.)

Example 14.

Let X_1, X_2*, and* X_3 *be independent standard normal random variables. If* $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2$*, and* $Y_3 = X_1 - X_3$*, compute the joint density function of* Y_1, Y_2, Y_3*.*

Solution. *Letting* $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2$, $Y_3 = X_1 - X_3$*, the Jacobian of these transformations is given by*

$$J = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3$$

As the preceding transformations yield that

$$X_1 = \frac{Y_1 + Y_2 + Y_3}{3} X_2 = \frac{Y_1 - 2Y_2 + Y_3}{3} X_3 = \frac{Y_1 + Y_2 - 2Y_3}{3}$$

we see from Equation (7.3) *that*

Example 4 Continued

$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)$

$$= \frac{1}{3} f_{X_1,X_2,X_3}\left(\frac{y_1 + y_2 + y_3}{3}, \frac{y_1 - 2y_2 + y_3}{3}, \frac{y_1 + y_2 - 2y_3}{3}\right)$$

Hence, as

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} e^{-\sum_{i=1}^{3} x_i^2/2}$$

we see that

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{3(2\pi)^{3/2}} e^{-Q(y_1,y_2,y_3)/2}$$

where $Q(y_1, y_2, y_3)$

$$= \left(\frac{y_1 + y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 - 2y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 + y_2 - 2y_3}{3}\right)^2$$

Example 4 Continued

$$= \frac{y_1^2}{3} + \frac{2}{3}y_2^2 + \frac{2}{3}y_3^2 - \frac{2}{3}y_2y_3$$

Example 15.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with rate λ. Let

$$Y_i = X_1 + \cdots + X_i \quad i = 1, \ldots, n.$$

(a) *Find the joint density function of $Y_1, \ldots, Y_n$.*

(b) *Use the result of part (a) to find the density of Y_n.*

Solution*(a) The Jacobian of the transformations* $Y_1 = X_1, Y_2 = X_1 + X_2, \ldots, Y_n = X_1 + \cdots + X_n$ *is*

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & & \cdots & & & \\ \cdots & & \cdots & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

Since only the first term of the determinant will be nonzero, we have $J = 1$. Now, the joint density function of $X_1, \ldots, X_n$ is given by

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \quad 0 < x_i < \infty, i = 1, \ldots, n$$

Hence, because the preceding transformations yield

$$X_1 = Y_1, X_2 = Y_2 - Y_1, \ldots, X_i = Y_i - Y_{i-1}, \ldots, X_n = Y_n - Y_{n-1}$$

Example 5 Continued

it follows from Equation (7.3) that the joint density function of $Y_1, \ldots, Y_n$ is $f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n - y_n)$

$$
\begin{aligned}
&= f_{X_1,\ldots,X_n}(y_1, y_2 - y_1, \ldots, y_i - y_{i-1}, \ldots, y_n - y_{n-1}) \\
&= \lambda^n \exp\left\{ -\lambda \left[y_1 + \sum_{i=2}^{n} (y_i - y_{i-1}) \right] \right\} \\
&= \lambda^n e^{-\lambda y_n} \quad 0 < y_1, 0 < y_i - y_{i-1}, i = 2, \ldots, n \\
&= \lambda^n e^{-\lambda y_n} \quad 0 < y_1 < y_2 < \cdots < y_n.
\end{aligned}
$$

(b) To obtain the marginal density of Y_n, let us integrate out the other variables one at a time. Doing this gives

$$
\begin{aligned}
f_{y_2,\ldots,Y_n}(y_2, \ldots, y_n) &= \int_0^{y_2} \lambda^n e^{-\lambda y_n} dy_1 \\
&= \lambda^n y_2 e^{-\lambda y_n} \quad 0 < y_2 < y_3 < \cdots < y_n.
\end{aligned}
$$

Example 5 Continued

Continuing, we obtain

$$
\begin{aligned}
f_{y_3,\ldots,Y_n}(Y_3,\ldots,y_n) &= \int_0^{y_3} \lambda^n y_2 e^{-\lambda y_n} dy_2 \\
&= \lambda^n \frac{y_3^2}{2} e^{-\lambda y_n} \quad 0 < y_3 < y_4 < \cdots < y_n.
\end{aligned}
$$

The next integration yields

$$
f_{Y_4,\ldots,Y_n}(y_4,\ldots,y_n) = \lambda^n \frac{y_4^3}{3!} e^{-\lambda y_n} \quad 0 < y_4 < \cdots < y_n.
$$

Continuing in this fashion gives

$$
f_{Y_n}(y_n) = \lambda^n \frac{y_n^{n-1}}{(n-1)!} e^{-\lambda y_n} \quad 0 < y_n.
$$

Example 5 Continued

which, in agreement with the result obtained in Example $3b$, shows that $X_1 + \cdots + X_n$ is a gamma random variable with parameters n and λ

Exercise 16.

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2;

(b) X_1, X_2, X_3.

Solution: (a)

$$p(0,0) = \frac{8.7}{13.12} = 14/39$$
$$p(0,1) = p(1,0) = \frac{8.5}{13.12} = 10/39$$
$$p(1,1) = \frac{5.4}{13.12} = 5/39$$

Exercise 6 Solution Continued

(b)

$$\begin{aligned}
p(0,0,0) =& \frac{8.7.6}{13.12.11} = 28/143 \\
p(0,0,1) =& p(0,1,0) = p(1,0,0) = \frac{8.7.5}{13.12.11} = 70/429 \\
p(0,1,1) =& p(1,0,1) = p(1,1,0) = \frac{8.5.4}{13.12.11} = 40/429 \\
p(1,1,1) =& \frac{5.4.3}{13.12.11} = 5/143
\end{aligned}$$

Exercise 17.

Repeat Problem 2 when the ball selected is replaced in the urn before the next selection.

Solution:

(a) $p(0,0) = (8/13)^2, p(0,1) = p(1,0) = (5/13)(8/13), p(1,1) = (5/13)^2$

(b)

$$p(0,0,0) = (8/13)^3$$
$$p(i,j,k) = (8/13)^2(5/13) \textit{if } i+j+k = 1$$
$$p(i,j,k) = (8/13)(5/13)^2 \textit{if } i+j+k = 2$$

Exercise 18.

The joint probability density function of X *and* Y *is given by*
$f(x,y) = c(y^2 - x^2)e^{-y} \quad -y \leq x \leq y, 0 < y < \infty$

(a) *Find* C.

(b) *Find the marginal densitites of* X *and* Y.

(c) *Find* $E[X]$

Solution:

$$f_Y(y) = c\int_{-y}^{y}(y^2 - x^2)e^{-y}dx$$

$$= \frac{4}{3}ey^3e^{-y}, -0 < y < \infty$$

$$\int_0^\infty f_Y(y)dy = 1 \Rightarrow c = 1/8 and\ so\ f_Y(y) = \frac{y^3e^{-y}}{6}, 0 < y < \infty$$

Exercise 8 solution continued

$$f_X(x) = \frac{1}{8}\int\limits_{|x|}^{\infty}(y^2 - x^2)e^{-y}dy$$

$$= \frac{1}{4}e^{-|x|}(1 + |x|) \text{upon using} - \int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$$

Exercise 19.

The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad 0 < x < 1, 0 < y < 2$$

(a) *Verify that this is indeed a joint density function.*

(b) *Compute the density function of X.*

(c) *Find $P\{X > Y\}$.*

(d) *Find $P\{Y > \frac{1}{2} | X < \frac{1}{2}\}$.*

(e) *Find $E[X]$.*

(f) *Find $E[Y]$.*

Exercise 9 solution

Solution:

(b) $f_X(x) = \frac{6}{7}\int_0^2 \left(x^2 + \frac{xy}{2}\right) dy = \frac{6}{7}(2x^2 + x)$

(c) $P\{X > Y\} = \frac{6}{7}\int_0^1 \int_0^x \left(x^2 + \frac{xy}{2} dydx\right) = \frac{15}{56}$

(d) $P\{y > 1/2 | X < 1/2\} = P\{Y > 1/2, X < 1/2\}/P\{X < 1/2\}$

$$= \frac{\int_{1/2}^2 \int_0^{1/2} \left(x^2 + \frac{xy}{2} dxdy\right)}{\int_0^{1/2} (2x^2 + x) dx}$$

Exercise 20.

An ambulance travels back and forth at a constant speed along a road of length L. At a certain moment of time, an accident occurs at a point uniformly distributed on the road. [That is, the distance of the point from one of the fixed ends of the road is uniformly distributed over $(0, L)$.] Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, and assuming independence of the variables, compute the distribution of the distance of the ambulance from the accident.

Solution :

Let X and Y denoted respectively the locations of the ambulance and the accident of the moment the accident occurs.

$$P\{|Y - X| < a\} = P\{Y < X < Y + a\} + P\{X < Y < X + a\}$$

Exercise 10 Solution Continued

$$= \frac{2}{L^2} \int_0^{L} \int_y^{\min(y+a,L)} dxdy$$

$$= \frac{2}{L^2} \left[\int_0^{L-a} \int_y^{y+a} dxdy + \int_{L-a}^{L} \int_y^{L} dxdy \right]$$

$$= 1 - \frac{L-a}{L} + \frac{a}{L^2}(L-a) = \frac{a}{L}\left(2 - \frac{a}{L}\right), 0 < a < L$$

Exercise

Exercise 21.

The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if, for some constant c, its joint density is

$$f(x,y) = \begin{cases} c & \text{if } (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

(a) *Show that $1/c =$ area of region R. Suppose that (X, Y) is uniformly distributed over the square centered at $(0,0)$ and with sides of length 2.*

(b) *Show that X and Y are independent, with each being distributed uniformly over $(-1,1)$.*

(c) *What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \leq 1\}$.*

Exercise 11 Solution

Solution:

(a) $1 = \iint f(x,y)dydx = \iint\limits_{(x,y)\in R} cdydx = cA(R)$
where $A(R)$ is the area of the region R.

(b)

$$f(x,y) = 1/4, -1 \le x, y \le 1$$
$$= f(x)f(y)$$
$$\text{where} f(v) = 1/2, -1 \le v \le 1.$$

(c) $P\{X^2 + Y^2 \le 1\} = \frac{1}{4} \iint\limits_{c} dydx = (\text{area of circle})/4 = \pi/4$

Exercise 22.

Suppose that n points are independently chosen at random on the circumference of a circle, and we want the probability that they all lie in some semicircle. That is, we want the probability that there is a line passing through the center of the circle such that all the points are on one side of that line, as shown in the following diagram:

image

Let $P_1, \ldots, P_n$ *denote the n points. Let A denote the event that all the points are contained in some semicircle, and let* A_i *be the event that all the points lie in the semicircle beginning at the point* P_i *and going clockwise for* $180^\circ, i = 1, \ldots, n$.

(a) *Express A in terms of the* A_i.

(b) *Are the* A_i *mutually exclusive?*

(c) *Find* $P(A)$.

Exercise 12 solution

Solution

(a) $A = \cup A_i$,

(b) yes

(c) $P(A) = \sum P(A_i) = n(1/2)^{n-1}$

Exercise 23.

Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$.] Find the probability that the distance between the two points is greater than $L/3$.

Solution:

$$P\{Y - X > L/3\} = \iint\limits_{\substack{y-x>L/3 \\ \frac{L}{2}<y<L \\ 0<x<\frac{L}{2}}} \frac{4}{L^2} dydx$$

Exercise 13 solution continued

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dydx + \int_{L/6}^{L/2}\int_{x+L/3}^{L} dydx\right]$$

$$= \frac{4}{L^2}\left[\frac{L^2}{12} + \frac{5L^2}{24} - \frac{7L^2}{72}\right] = 7/9$$

Exercise 24.

Let $f(x,y) = 24xy \quad 0 \leq x \leq 1, \quad 0 \leq 1, \quad 0 \leq x+y \leq 1$ *and let it equal* 0 *otherwise.*

1. *Show that* $f(x,y)$ *is a joint probability density function.*
2. *Find* $E[X]$*.*
3. *Find* $E[Y]$*.*

Solution:

(a) *We must show that* $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$. *Now,*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^1\int_0^{1-y} 24xy \; dxdy$$
$$= \int_0^1 12y(1-y)^2 dy$$

Exercise 14 solution continued

$$= \int_0^1 12(y - 2y^2 + y^3)dy$$
$$= 12(1/2 - 2/3 + 1/4) = 1.$$

(b)

$$E[X] = \int_0^1 x f_X(x) dx$$
$$= \int_0^1 x \int_0^{1-x} 24 dy dx$$
$$= \int_0^1 12x^2(1-x)^2 dx = 2/5$$

(c) 2/5

Exercise 25.

The random variables X and Y have joint density function

$$f(x, y) = 12xy(1 - x) \quad 0 < x < 1, 0 < y < 1$$

and equal to 0 *otherwise.*

(a) *Are X and Y independent?*

(b) *Find* $E[X]$.

(c) *Find* $E[Y]$.

(d) *Find* $Var(X)$.

(e) *Find* $Var(Y)$.

Exercise 15 Soltuion

Solution:

(a) yes

$$f_X(x) = 12x(1-x)\int_0^1 y dy = 6x(1-x), 0 < x < 1$$

$$f_Y(y) = 12y\int_0^1 x(1-x)dx = 2y, 0 < y < 1$$

(b) $E[X] = \int_0^1 6x^2(1-x)dx = 1/2$

(c) $E[Y] = \int_0^1 2y^2 dy = 2/3$

(d) $Var(X) = \int_0^1 6x^3(1-x)dx - 1/4 = 1/20$

(e) $Var(Y) = \int_0^1 2y^3 dy - 4/9 = 1/18$

Exercise 26.

The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that an article of 10 *pages contains* (a)0 *and* (b)2 *or more typographical errors? Explain your reasoning!*

Solution :

(a) e^{-2}

(b) $1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$

The number of typographical errors on each page should approximately be Poisson distributed and the sum of independent Poisson random variables is also a Poisson random variable.

Exercise 27.

The joint density of X and Y is

$$f(x,y) = c(x^2 - y^2)e^{-x}, 0 \leq x < \infty, -x \leq y \leq x.$$

Find the conditional distribution of Y, given X = x.

Solution:

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{(x^2 - y^2)e^{-x}}{\int_{-x}^{x}(x^2 - y^2)e^{-x}dx} \\ &= \frac{3}{4x^3}(x^2 - y^2), -x < y < x \end{aligned}$$

$$\begin{aligned} F_{Y|X}(y|x) &= \frac{3}{4x^3}\int_{-x}^{y}(x^2 - y^2)dy \\ &= \frac{3}{4x^3}(x^2y - y^3/3 + 2x^3/3), -x < y < x \end{aligned}$$

Exercise 28.

If 3 trucks break down at points randomly distributed on a road of length L, find the probability that no 2 of the trucks are within a distance d of each other when $d \leq L/2$.

Solution:

$$\left(\frac{L-2d}{L}\right)^3$$

Exercise 29.

If X and Y are independent random variables both uniformly distributed over $(0,1)$, find the joint density function of
$R = \sqrt{x^2 + Y^2}, \Theta = \tan^{-1} Y/X$.

Solution :

$$f_{R,\theta}(r,\theta), \quad 0 < r\sin\theta < 1, \quad 0 < r\cos\theta < 1, \quad 0 < \theta < \pi/2, \quad 0 < r < \sqrt{2}$$

Exercise 30.

Suppose that $X_i, i = 1, 2, 3$ are independent Poisson random variables with respective means $X_i, i = 1, 2, 3$. Let $X = X_1 + X_2$ and $Y = X_2 + X_3$. The random vector X, Y is said to have a bivariate Poisson distribution. Find its joint probability mass function. That is, find $P\{X = n, Y = m\}$.

Solution:

$$P(X = n, Y = m) = \sum_i P(X = n, Y = m | X_2 = i) P(X_2 = i)$$

$$= e^{-(\lambda_1 + \lambda_2 + \lambda_3)} \sum_{i=0}^{\min(n,m)} \frac{\lambda_1^{n-i}}{(n-1)} \frac{\lambda^{m-i}}{(m-i)!} \frac{\lambda_2^i}{i!}$$

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